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Exact Solution for Dynamic Oscillations of Re-Entry Bodies

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Nomenclature

A	= reference area of re-entry body
a	= parameter in confluent hypergeometric function, Eq. (9)
C_D, C_L, C_m	= standard aerodynamic drag, lift and moment coefficients
$C_{L_\alpha} = \dfrac{\partial C_L}{\partial \alpha}$	= aerodynamic lift coefficient slope vs angle of attack
$C_{m_\alpha} = \dfrac{\partial C_m}{\partial \alpha}$	= aerodynamic moment coefficient slope vs angle of attack
$C_{m_q} = \dfrac{\partial C_m}{\partial (qL/v)}$	= damping moment ($C_{m_q} < 0$) due to pitching angular velocity
C_{N_α}	= $C_D + C_{L_\alpha}$
${}_1F_1$	= confluent hypergeometric function, Eq. (10)
I	= mass moment of inertia in pitch
J_0	= zero order Bessel function of the first kind
k_1, k_2, k_3	= Allen's aerodynamic stability parameters, Eqs. (3, 4, and 5)
L	= characteristic reference length of re-entry body
m	= mass of re-entry body
q	= angular velocity in pitch relative to Earth
v	= velocity along trajectory of re-entry body
y	= altitude from reference level where $\rho = \rho_0$ when $y = 0$, Eq. (1)
$Y = \beta y$	= nondimensional altitude
$Z = Z_0 e^{-Y}$	= variable replacing nondimensional altitude, Eq. (8)
α	= angle of attack of re-entry body relative to its trajectory
β	= atmospheric density exponential factor, Eq. (1)
δ_0	= nondimensional mass ratio, Eq. (6)
$\theta_E = -\gamma_0 > 0$	= downward angle between straight line re-entry flight path and Earth's surface
ρ	= atmospheric air density

FRIEDRICH and Dore[1] and Allen[2] independently obtained a Bessel's function solution that approximated the dynamic oscillations of a hypersonic re-entry body that is descending through a planetary atmosphere. In order to establish the range of validity of this Bessel function approximation we have derived an exact solution of Allen's differential equation for a straight line trajectory. This exact solution is a special case of the confluent hypergeometric function that allows the appropriate criteria for dynamic stability to be formulated explicitly in terms of the basic aerodynamic coefficients. The Bessel function approximation is shown to provide an excellent solution for the linearized oscillations of the usual ballistic missile. The exact solution is found to be necessary only when one considers re-entry bodies which have such a small aerodynamic restoring moment that the special case of near critical damping is approached. This condition of near critical damping is studied, and the relations between the aerodynamic coefficients that will achieve this condition are derived.

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Allen's Differential Equation and its Exact Solution

Allen[2] derived the linearized differential equation for the angle-of-attack oscillations of a rotationally symmetric re-entry body that had a sufficiently large drag so that the descending flight path was essentially a straight line. He also assumed that the aerodynamic force and moment coefficients remained constant, so his analysis would be most applicable to the hypersonic case. The altitude range to be considered was such that the acceleration due to gravity could be considered to be a constant independent of altitude, and the variation of the Earth's atmospheric density could be approximated by the exponential function

$$\rho/\rho_0 = e^{-\beta y} = e^{-Y}, \quad \beta^{-1} \approx 22{,}000 \text{ ft} \approx 6710\,\text{m} \tag{1}$$

Under these assumptions Allen obtained the following differential equation, where the angle of attack variation (α) may be considered to be in either pitch, yaw or any vector sum of these two angular displacements

$$d^2\alpha/dY^2 + 2k_1 e^{-Y} d\alpha/dY + (k_2 e^{-Y} + k_3 e^{-2Y})\alpha = 0 \tag{2}$$

$$k_1 = (\delta_0/2)[C_D - C_{L_\alpha} + (C_{m_q} + C_{m_{\dot\alpha}})(mL^2/I)] \tag{3}$$

$$k_2 = \delta_0[C_{L_\alpha} - C_{m_\alpha}(mL^2/I)(\beta L \sin\theta_E)^{-1}] \tag{4}$$

$$k_3 = \delta_0{}^2 C_{L_\alpha}[-C_D - C_{m_q}(mL^2/I)] \tag{5}$$

These are in Allen's[2] notation except that we have replaced the constant Allen designated as k_0 by

$$\delta_0 = \rho_0 A(2\beta m \sin\theta_E)^{-1} = k_0/2C_D \tag{6}$$

We have found that the exact solution of Eq. (2) could be written as

$$\alpha(Y) = \alpha_0\, {}_1F_1(a, 1, Z)\exp[(k_1/Z_0) - (\tfrac{1}{2})]Z \tag{7}$$

$$Z(Y) = 2(k_1{}^2 - k_3)^{1/2} e^{-Y} = Z_0 e^{-Y} \tag{8}$$

$$a = (\tfrac{1}{2}) - (k_1 + k_2)Z_0{}^{-1} \tag{9}$$

where

$${}_1F_1(a, 1, Z) = 1 + aZ + a(a+1)(2!)^{-2}Z^2 + a(a+1)(a+2)(3!)^{-2}Z^3 + \ldots \tag{10}$$

is a special case of the confluent hypergeometric function, e.g. see Slater.[3] This function is an infinite series except when a is a negative integer so that $a = -n$, and in this case ${}_1F_1$ reduces to a finite polynomial of order n. For values of $-a > 10$ we can replace this confluent hypergeometric function by its asymptotic form when $a \to -\infty$ in terms of the zero-order Bessel function (J_0) of the first kind as follows:

$${}_1F_1(a, 1, Z) \approx e^{Z/2} J_0\{[(2-4a)Z]^{1/2}\} \tag{11}$$

It will be shown that the parameter a defines the number of oscillations that will occur. For example, when $a = -n$ is a negative integer, then the exact solution given by Eq. (7) cannot have more than n oscillations. Consequently, small negative values of a yield a motion that resembles the motion produced by near critical damping in oscillating systems. However, the usual ballistic missile has $-a > 10$ and in these cases Eq. (11) provides an excellent approximation that reduces Eq. (7) to

$$\alpha(Y) = \alpha_0 J_0\{[(2-4a)Z]^{1/2}\}\exp(k_1 Z/Z_0) \tag{12}$$

This is identical to Allen's[2] approximate solution to Eq. (2), and it will be shown that it does provide an excellent solution for the dynamic oscillations of a typical ballistic missile, as predicted by Allen.

Stability Parameter in Terms of the Aerodynamic Coefficients

If we now follow Allen[2] and neglect the term $C_{m_{\dot{\alpha}}}$ in Eq. (3), a justifiable assumption at high Mach numbers since this coefficient is identically zero in Newtonian hypersonic flow, then Eqs. (3, 5, and 8) yield the following:

$$(Z_0/2)^2 = (k_1{}^2 - k_3) = (\delta_0/2)^2[C_D + C_{L_\alpha} + C_{m_q}(mL^2/I)]^2 \geqq 0 \quad (13)$$

In this case Eq. (7) always yields a real solution and we find that there are three possible cases, as follows:

Case I when

$$(-C_{m_q})(mL^2/I) > (C_D + C_{L_\alpha}) = C_{N_\alpha} > 0 \quad (14)$$

$$\alpha_1(Y) = \alpha_0\, {}_1F_1(a_1, 1, Z)\exp\{-\delta_0[-C_{m_q}(mL^2/I) - C_D]e^{-Y}\} \quad (15)$$

$$Z(Y) = \delta_0[-C_{m_q}(mL^2/I) - C_{N_\alpha}]e^{-Y} = Z_0 e^{-Y} > 0 \quad (16)$$

$$a_1 = 1 + C_{m_\alpha}(mL^2/I)(\beta L \sin\theta_E)^{-1}[-C_{m_q}(mL^2/I) - C_{N_\alpha}]^{-1} \quad (17)$$

Case II when

$$C_{N_\alpha} = (C_D + C_{L_\alpha}) > (-C_{m_q})(mL^2/I) \quad (18)$$

$$\alpha_2(Y) = \alpha_0\, {}_1F_1(a_2, 1, Z)\exp(-\delta_0 C_{L_\alpha})e^{-Y} \quad (19)$$

$$Z(Y) = \delta_0[C_{N_\alpha} + C_{m_q}(mL^2/I)]e^{-Y} = Z_0 e^{-Y} > 0 \quad (20)$$

$$a_2 = C_{m_\alpha}(mL^2/I)(\beta L \sin\theta_E)^{-1}[C_{N_\alpha} + C_{m_q}(mL^2/I)]^{-1} \quad (21)$$

Case III when

$$[C_{N_\alpha} + C_{m_q}(mL^2/I)] = 0 = (2/\delta_0)(k_1{}^2 - k_3)^{1/2} \quad (22)$$

$$\alpha_3(Y) = \alpha_0 J_0([4\delta_0(-C_{m_\alpha})(mL^2/I)(\beta L \sin\theta_E)^{-1}e^{-Y}]^{1/2}) \times \exp(-\delta_0 C_{L_\alpha})e^{-Y} \quad (23)$$

Case III is a singular solution corresponding to $Z_0 = 0$ in Eq. (8), and consequently $a \to -\infty$ so that either Eq. (11) must be introduced in a limiting process, or the original differential Eq. (2) must be solved as by Allen,[2] using the relation $(k_1{}^2 - k_3) = 0$. Consequently, Allen's approximate solution is exact for case III, but as will be shown, it also provides an excellent approximation to the exact solutions given by Eqs. (15) and (19) as long as $-a > 10$.

Application to a Ballistic Missile

The blunt nose ballistic missile usually corresponds to case I as defined by Eq. (14) because its C_{L_α} is either very small or negative. For example, any conical nose having a semivertex angle greater than 45° has $C_{L_\alpha} < 0$ in hypersonic flow. In this case the damping of the angle of attack oscillations can only be produced by $C_{m_q} < 0$. This real damping occurs for case I even though $k_1 = 0$ as may be shown by introducing Eq. (3) into Eqs. (15) and (16) so as to obtain

$$\alpha_1(Y) = \alpha_0\, {}_1F_1(a_1, 1, Z)e^{-Z/2} \quad (24)$$

$$Z(Y) = 2\delta_0(-C_{L_\alpha})e^{-Y} = 2\delta_0[-C_{m_q}(mL^2/I) - C_D]e^{-Y} > 0 \quad (25)$$

since we must now have

$$C_{m_q}(mL^2/I) + C_D = C_{L_\alpha} < 0 \quad (26)$$

in order to satisfy both $k_1 = 0$ and Eq. (14) for case I.

Although most ballistic missiles have $k_1 < 0$, as shown by Allen,[2] still the exact solution given by Eq. (7) shows that even positive values of k_1 can have real damping as long as $k_1 < Z_0/2$ and $a < 0$. However, this is not true for $a > 0$ because ${}_1F_1$ then increases exponentially with Z; whereas, for $a < 0$ this confluent hypergeometric function behaves as a simple polynomial of order N defined by $(-a+1) > N \geqq (-a)$, with all of its roots contained in the interval $0 < Z < -4a$, e.g. see Slater.[3] Consequently, the parameter a, which depends primarily upon C_{m_α}, determines the number of oscillations while the term k_1 provides the exponential damping. This behavior is also evident in the approximate solution of Allen, or Eq. (12), which is valid for large negative values of the constant a.

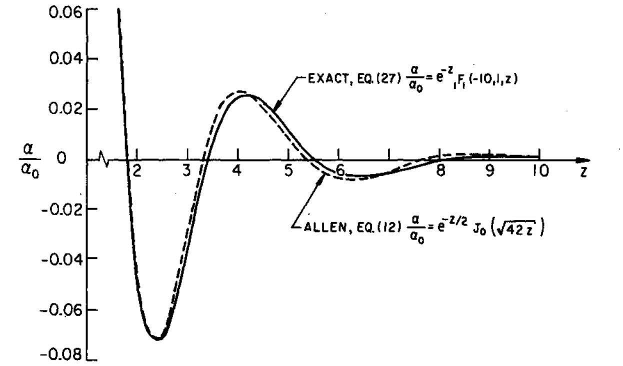

Fig. 1 Comparison of Allen's approximate solution with the exact Eq. (27) when $C_{L_\alpha} = 0$, $a = -10$, $k_1 = -Z_0/2$, $k_2 = 11Z_0$ and $k_3 = 0$.

As shown by Allen[2] for a typical ballistic missile a in Eq. (9) is primarily determined by k_2, Eq. (4), as long as $C_{m_\alpha} \approx -1$ since the term $(\beta L \sin\theta_E)^{-1} \approx 10^3$. In this case the parameter $a \approx -10^3$ so that Eq. (12) provides an excellent approximation to Eq. (7). However, since C_{m_α} can be made arbitrarily small by simply moving the center of gravity of the missile aft towards the neutral point, it is worthwhile to investigate the effect of having $0 > a > -10$ upon the dynamic oscillations governed by Allen's Eq. (2). Let us now consider the case having $C_{L_\alpha} = 0 = k_3$. Then Eqs. (7) and (8) reduce to the following:

$$\alpha(Y) = \alpha_0\, {}_1F_1(a, 1, Z)e^{-Z} \quad (27)$$

$$Z(Y) = 2|k_1|e^{-Y} = Z_0 e^{-Y} > 0 \quad (28)$$

Since we are considering a missile having $C_{L_\alpha} = 0$, we are restricted to case I if we desire a damped oscillation; consequently, $k_1 < 0$ from Eq. (3), and Eq. (9) reduces to

$$a = 1 - (k_2/Z_0) = 1 - k_2/(-2k_1) \quad (29)$$

The exact solution given by Eq. (27) is compared with Allen's approximate solution, Eq. (12), for the case of $a = -10$ in Fig. 1. The oscillations through the first two roots are not shown in Fig. 1 because even on this expanded scale, which originally starts at $\alpha/\alpha_0 = 1$, the two curves are essentially the same for $Z < 1$. It should also be noted that $Z = 0$ corresponds to $Y \to \infty$, and the reference level $Y = 0$ where $\rho = \rho_0$ corresponds to $Z = Z_0 = -2k_1$. Therefore the numerical value of k_1 determines the number of oscillations that will be encountered. When $a = -10$ Eq. (27) has ten roots but only six are encountered for $Z < 10$, and as shown by Allen's[2] calculations the typical ballistic missile would have $Z_0 < 10$.

The effect of varying k_1 is shown in Fig. 2 for the case of $a = -1$. Here $-C_{m_\alpha} \approx 10^{-3}$ so that the restoring moment is so

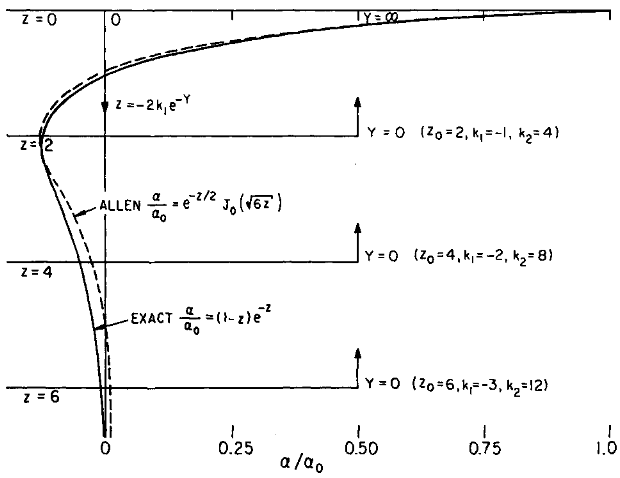

Fig. 2 Comparison of Allen's approximate solution with the exact Eq. (27) when $C_{L_\alpha} = 0$, $a = -1$, $k_1 = -Z_0/2$, $k_2 = 2Z_0$ and $k_3 = 0$.

small that the exact solution, Eq. (27), has only one root at $Z = 1$. Consequently, the angle of attack variation is the same as that of a critically damped dynamical system as indicated in Fig. 2. The change in reference level ($Y = 0$) is given for several values of k_1. For $Z > 6$ the angle of attack essentially remains zero for the case shown in Fig. 2, and the exact solution given by Eq. (27) for $a = -1$ has no oscillations for $Z > 1$.

It is now evident that for $-a > 10$ Allen's approximate solution, as given by Eq. (12), provides an excellent approximation to his differential equation for the straight line descent of a large drag ballistic missile, Eq. (2). It is necessary to introduce the exact solution, Eq. (7), only when $-a < 10$, a condition which can be attained only by having a sufficiently small C_{m_α} so that $-C_{m_\alpha}(\beta L \sin\theta_E)^{-1} < 10$. In this case the oscillation resembles that of a dynamical system that is near critical damping.

Stone[4] has derived a differential equation similar to Allen's that includes the effect of a constant rate of roll. However, Stone's differential equation for zero roll rate corresponds to $k_3 = 0$ in our Eq. (2), and the omission of C_{L_α} in k_2, Eq. (4). Consequently, Stone's solution for a missile descending without roll would only correspond to the particular solution given by Eq. (27) whenever $C_{L_\alpha} \approx 0$ and $-a > 10$.

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Equivalence of the Minimum Norm and Gradient Projection Constrained Optimization Techniques

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THE title "direct" optimization techniques are shown to be both theoretically and computationally equivalent in the common event that the former is employed as a constrained optimization procedure. The parametric and functional versions of these formulations have heretofore been considered alternative mathematical procedures for iterative solutions of trajectory optimization problems and other problems of similar structure. The reported difference in computational behavior of algorithms based upon the two formulations is traced to inadequate knowledge of the mathematical relationship of the methods and inconsistent computer programming. The results given herein can be employed to eliminate redundancy in existing computer programs and will generally promote increased understanding of these often-used processes.

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Parameter Space Results

Two well-known optimization formulations are studied; these techniques determine values for each of n unknown parameters $\{p_1, p_2, \ldots, p_n\}$ such that a given payoff function

$$\phi = \phi(p_1, p_2, \ldots, p_n) \tag{1}$$

assumes an extreme value, subject to the satisfaction of m equality constraints

$$\psi_j(p_1, p_2, \ldots, p_n) = 0; \quad j = 1, 2, \ldots, m; \quad m < n \tag{2}$$

The functions (1) and (2) are generally nonlinear and typically require numerical integration for their evaluation.

Many numerical techniques have been developed and used in solving a variety of problems of the above type and its function space generalization.[1,2] Among the more popular techniques for solving trajectory optimization problems of the above form are the Gradient Projection and Minimum Norm Optimization techniques. These formulations are based upon local linearizations of the payoff and constraint functions as

$$\Delta\phi = G^T \Delta P \tag{3}$$

and

$$\Delta\Psi - A\Delta P = 0 \tag{4}$$

where $\Delta\phi$ = objective (linearly predicted) charge in the payoff function, $\Delta\Psi = m \times 1$ matrix of constraint violations ("objective minus current"), $G^T = [\partial\phi/\partial p_1 \ldots \partial\phi/\partial p_n]$ = locally evaluated payoff function gradient, $A = [\partial\psi_j/\partial p_i] = m \times n$ locally evaluated constraint Jacobian, and $\Delta P^T = [\Delta p_1 \ldots \Delta p_n]$ = matrix of parameter corrections. The gradient projection formulation determines corrections (ΔP) which locally extremizes the predicted improvement Eq. (3) in ϕ subject to Eq. (4) (first order satisfaction of the constraints) and a restriction upon the norm of the corrections as measured by

$$\Delta s^2 = \Delta P^T W \Delta P \tag{5}$$

where Δs^2 is an assigned value and W is an $n \times n$ positive definite weighting matrix. The minimum norm optimization formulation determines corrections which minimize the control variable norm (Δs^2) required to satisfy the constraints of Eq. (4) and yield a specified (predicted) improvement of Eq. (3) in the payoff function.

Gradient projection Solution

Since this solution, as well as the minimum norm solution, has been fully derived in the literature,[1-5] only the key results are given here. It is the relationship between known results, not derivation of these results, which is being studied. Formally, the correction matrix (ΔP) is sought which extremizes the predicted improvement of Eq. (3) in the payoff function, subject to satisfaction of Eqs. (4) and (5). The desired corrections have been determined[3-5] to be

$$\Delta P = W^{-1}A^T(AW^{-1}A^T)^{-1}\Delta\Psi \pm \left(\frac{\Delta s^2 - \Delta\Psi^T(AW^{-1}A^T)^{-1}\Delta\Psi}{G^TW^{-1}G - G^TW^{-1}A^T(AW^{-1}A^T)^{-1}AW^{-1}G}\right)^{1/2} \times [W^{-1}G - W^{-1}A^T(AW^{-1}A^T)^{-1}AW^{-1}G] \tag{6}$$

where the positive (negative) sign has been shown[5] to yield a local maximum (minimum) improvement ($\Delta\phi$). The various conditions under which the scalar numerator and denominator under the square root sign are positive, negative, or zero are given by Junkins[5]; only the most interesting case [when $\Delta s^2 = \Delta\Psi^T(AW^{-1}A^T)^{-1}\Delta\Psi$] will be discussed here, after development of the minimum norm solution for comparison.

Minimum norm solution for "constraint-only" case

The minimum norm solution was originally designed for solution of underdetermined boundary constraint problems, with no explicit treatment of the payoff function. As has been accomplished by several investigators,[3-5] however, the minimum norm solution can be readily adapted to solve constrained optimization problems. This is accomplished by restructuring a constrained